HOMOLOGICAL INDEX FOR 1-FORMS AND A MILNOR NUMBER FOR ISOLATED SINGULARITIES

W. EBELING, S. M. GUSEIN-ZADE, AND J. SEADE

ABSTRACT. We introduce a notion of a homological index of a holomorphic 1-form on a germ of a complex analytic variety with an isolated singularity, inspired by X. Gómez-Mont and G.-M. Greuel. For isolated complete intersection singularities it coincides with the index defined earlier by two of the authors. Subtracting from this index another one, called radial, we get an invariant of the singularity which does not depend on the 1-form. For isolated complete intersection singularities this invariant coincides with the Milnor number. We compute this invariant for arbitrary curve singularities and compare it with the Milnor number introduced by R.-O. Buchweitz and G.-M. Greuel for such singularities.

1. INTRODUCTION

Indices of vector fields on singular varieties have been studied in a number of papers, e.g., [2, 10, 15, 19, 20, 21]. One problem with this approach is that the conditions for a vector field on the ambient space to be tangent to a singular space are rather strong and this makes their study difficult. On the contrary a 1-form on the ambient space always defines a 1-form on a subvariety. This and other facts motivated the study of indices of 1-forms on singular varieties in [6, 7].

Here we introduce a notion of a homological index of a 1-form on a complex analytic variety with an isolated singular point analogous to the homological index for vector fields defined in [10]. Due to results of [12], the homological index of a 1-form on an isolated complete intersection singularity is identified with the index defined in [6, 7]. For a 1-form on an isolated complex analytic singularity there is defined another index, which we call radial. This index, as well as the homological one, satisfy the law of conservation of number. For the homological index this follows from [9]. For 1-forms on smooth varieties both indices are equal to the usual Poincaré–Hopf index. This implies that the difference of these two indices does not depend on the 1-form and is an invariant of the singularity. For isolated complete intersection singularities we identify it with the Milnor number. It is natural to try to compare the invariant of an isolated singular point of a variety introduced in this paper with other invariants coinciding with the Milnor number for isolated complete intersection singularities.

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For curve singularities a notion of the Milnor number was introduced by R.-O. Buchweitz and G.-M. Greuel in [4]. For a smoothable curve singularity $(C, 0)$ it is equal to $1 - \chi(\widetilde{C})$ for a smoothing $\widetilde{C}$ of the singularity, $\chi(\cdot)$ is the Euler characteristic. However, there exist surface singularities which have smoothings with different Euler characteristics ([18]). This does not permit to generalize the notion of the Milnor number to higher dimensions so that for smoothable singularities it has the usual expression in terms of the Euler characteristic. We compute our invariant for curve singularities and compare it with the Milnor number of R.-O. Buchweitz and G.-M. Greuel.

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2. Indices of 1-forms

In [7] there was defined a notion of an index of a 1-form on a (real) manifold with isolated singularities. Here we shall consider complex analytic varieties with isolated singularities and (complex) 1-forms on them. Therefore we reformulate the definition from [7] in this setting.

Let $(V, 0) \subset (\mathbb{C}^N, 0)$ be a germ of a purely n-dimensional complex analytic variety with an isolated singularity at the origin 0. Let ω be a 1-form on V with an isolated singularity at the origin 0. Here this means that ω is a continuous, nowhere-vanishing section of the complex cotangent bundle of $V - \{0\}$. We consider an index that measures the "lack of radiality" of such a 1-form. This is similar to the index for vector fields considered in [2, 5, 15, 21].

Let us fix a radial vector field v_{rad} on $(V, 0)$, e.g., the gradient on the smooth part of V of the real valued function $\|z\|$ with respect to a Riemannian metric. A radial 1-form ω_{rad} is a (complex) 1-form on V whose value on the radial vector field v_{rad} has positive real part at each point in a punctured neighbourhood of the origin 0 in V. The space of such 1-forms is connected.

Let ω_1 and ω_2 be 1-forms on $(V, 0)$ with isolated singularities at the origin. Choose $\varepsilon > \varepsilon' > 0$ sufficiently small, let $K_\varepsilon = V \cap S_\varepsilon$ and $K_{\varepsilon'} = V \cap S_{\varepsilon'}$ be the corresponding links, and let Z be the cylinder $V \cap [B_\varepsilon \setminus \mathrm{Int}(B_{\varepsilon'})]$, where B_ρ is the ball of radius ρ around the origin 0 in $\mathbb{C}^N$, S_ρ is its boundary. Let $\widetilde{\omega}$ be a 1-form on the cylinder Z which coincides with ω_1 in a neighbourhood of K_ε and with ω_2 in a neighbourhood of $K_{\varepsilon'}$ and which has isolated singular points $Q_1, ..., Q_s$ inside C. The sum $d(\omega_1, \omega_2)$ of the (usual) local indices $\mathrm{ind}_{Q_i} \widetilde{\omega}$ of the form $\widetilde{\omega}$ at these points depends only on the forms ω_1 and ω_2 and will be called *the difference* of these forms. One has $d(\omega_1, \omega_2) = -d(\omega_2, \omega_1)$.

Definition 2.1. The *radial index* $\mathrm{Ind}_{\mathrm{rad}}(\omega; V, 0)$ (or simply $\mathrm{Ind}_{\mathrm{rad}} \omega$) of the 1-form ω on V is defined by the equation

$$\mathrm{Ind}_{\mathrm{rad}} \omega = (-1)^n + d(\omega, \omega_{\mathrm{rad}}).$$

REMARK 2.2. One can see that the index of the radial 1-form ω_{rad} is equal to $(-1)^n$. The sign is chosen so that this index coincides with the usual one when V is smooth at 0.

REMARK 2.3. There is a one-to-one correspondence between complex 1-forms on a complex analytic manifold $V - \{0\}$ and real 1-forms on it. Namely, to a complex 1-form ω one associates the real 1-form $\eta = \mathrm{Re}\,\omega$; the 1-form ω can be restored from η by the formula $\omega(v) = \eta(v) - i\eta(iv)$ for $v \in T_x(V - \{0\})$. This explains why the radial index of a complex 1-form can be expressed through the corresponding index of its real part defined in [7]. In other words the radial index $\mathrm{Ind}_{\mathrm{rad}}\,\omega$ of a complex 1-form ω can be defined as $(-1)^n$-times the radial index of the real part of ω.

REMARK 2.4. The radial index obviously satisfies the law of conservation of number, i.e., if ω' is a 1-form on V close to ω, then:

$$\mathrm{Ind}_{\mathrm{rad}}(\omega; V, 0) = \mathrm{Ind}_{\mathrm{rad}}(\omega'; V, 0) + \sum \mathrm{Ind}_{\mathrm{rad}}(\omega'; V, x),$$

where the sum on the right hand side is over all those points x in a small punctured neighbourhood of the origin 0 in V where the form ω' vanishes (this follows from the fact that $d(\omega_1, \omega_3) = d(\omega_1, \omega_2) + d(\omega_2, \omega_3)$).

EXAMPLE 2.5. Let ω be a holomorphic 1-form on a curve singularity $(C, 0)$ with $C = \cup_{i=1}^r C_i$, where C_i are the irreducible components of C. Let t_i be a uniformization parameter on the component C_i and let the restriction $\omega_{|C_i}$ be of the form $(a_i t_i^{m_i} + \textit{terms of higher degree})dt_i$, $a_i \neq 0$. Then $\mathrm{Ind}_{\mathrm{rad}}\,\omega_{|C_i} = m_i$. Therefore $d(\omega_{|C_i}, \omega_{\mathrm{rad}|C_i}) = m_i + 1$, $d(\omega, \omega_{\mathrm{rad}}) = \sum_{i=1}^r (m_i + 1)$, $\mathrm{Ind}_{\mathrm{rad}}\,\omega = \sum_{i=1}^r m_i + (r - 1)$.

EXAMPLE 2.6. Let $(W, 0) \subset (\mathbb{R}^M, 0)$ be a germ of a (real) analytic variety with an isolated singular point at the origin and let $f : (W, 0) \to (R, 0)$ be a germ of a real analytic function on $(W, 0)$ with an isolated critical point at the origin (the last means that the differential df has an isolated singular point at the origin on V). One can show that the (radial) index of the differential df (as defined in [7] for real 1-forms) is equal to $1 - \chi(F_-)$, where F_- is the "negative" Milnor fibre $\{f = -\delta\} \cap B_\varepsilon$ for $0 < \delta \ll \varepsilon$ small enough. This is an analogue for singular varieties of a formula of [3]. Let f be a germ of a holomorphic function on a complex analytic isolated singularity $(V, 0)$ with an isolated critical point at the origin. The map f defines a Milnor fibration (as noticed, e.g., by H. Hamm [13]). One can show that the Milnor fibre of f is homotopy equivalent to the Milnor fibre of its real part $\mathrm{Re}\,f$. Therefore the radial index $\mathrm{Ind}_{\mathrm{rad}}\,df$ of its differential df is equal to $(-1)^n(1 - \chi(F))$, where F is the Milnor fibre $\{f = \delta\} \cap B_\varepsilon$ of the function f on V $(0 < |\delta| \ll \varepsilon$ are small enough).

Now let $(V, 0) \subset (\mathbb{C}^{n+k}, 0)$ be an isolated complete intersection singularity, defined by an analytic map

$$f = (f_1, ..., f_k) : (\mathbb{C}^{n+k}, 0) \to (\mathbb{C}^k, 0).$$

For a 1-form ω on V with an isolated singularity at the origin 0, the covectors ω, df_1, ..., df_k are linearly independent at each point of V away from 0. This defines a map $(\omega, df_1, ..., df_k)$ from $V \setminus \{0\}$ into the Stiefel manifold $W_{k+1}^*(\mathbb{C}^{n+k})$ of $(k + 1)$-frames in

the dual $\mathbb{C}^{n+k}$. The manifold $W_{k+1}^*(\mathbb{C}^{n+k})$ is $(2n-2)$-connected and its first non-zero homotopy group $\pi_{2n-1}(W_{k+1}^*(\mathbb{C}^{n+k})$ is isomorphic to $\mathbb{Z}$ (see, e.g., [14]). Therefore the restriction of this map to the link K_ε of $(V,0)$ has a degree: that of the map induced in the homology of dimension $(2n-1) = \dim K_\varepsilon$.

Definition 2.7. The *index* $\mathrm{Ind}(\omega; V, 0) = \mathrm{Ind}\,\omega$ of the 1-form ω is the degree of the map

$$(\omega, df_1, ..., df_k) : K_\varepsilon \to W_{k+1}^*(\mathbb{C}^{n+k}).$$

This index, defined in [6, 7], is an analogue of the GSV-index for vector fields, introduced in [11, 20]. It is proved in [7], Proposition 3, that this index equals the number of zeroes, counted with multiplicities, of any extension of ω to a Milnor fibre $V_t = f^{-1}(t) \cap B_\varepsilon$ of $(V, 0)$. Let $\mu(V, 0)$ be the Milnor number of the isolated complete intersection singularity $(V, 0)$.

Proposition 2.8. *For a 1-form ω on $(V, 0)$ with an isolated singularity at the origin 0 one has*

$$\mathrm{Ind}\,\omega = \mu(V, 0) + d(\omega, \omega_{\mathrm{rad}}) + (-1)^n$$

and therefore the difference $\mathrm{Ind}\,\omega - \mathrm{Ind}_{\mathrm{rad}}\,\omega$ between its index and its radial index is independent of ω and is equal to the Milnor number $\mu(V, 0)$.

An analogue of this proposition for vector fields also holds: [21], Proposition 1.4. If the 1-form ω is holomorphic, i.e., if it is the restriction to $(V, 0)$ of a holomorphic 1-form $\omega = \sum_{i=1}^{n+k} A_i(x)dx_i$ on $(\mathbb{C}^{n+k}, 0)$, this index has an algebraic expression as the dimension of a certain algebra [6, 7]. Namely,

$$\mathrm{Ind}\,\omega = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^N, 0}/I,$$

where I is the ideal generated by $f_1, ..., f_k$ and the $(k+1) \times (k+1)$-minors of the matrix:

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{n+k}} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_{n+k}} \\ A_1 & \cdots & A_{n+k} \end{pmatrix}.$$

This formula was obtained by Lê D.T. and G.-M. Greuel for the case when ω is the differential of a function ([12, 16]).

3. THE HOMOLOGICAL INDEX

Let $(V, 0) \subset (\mathbb{C}^N, 0)$ be any germ of an analytic variety of pure dimension n with an isolated singular point at the origin (not necessarily a complete intersection). Given a holomorphic form ω on $(V, 0)$ with an isolated singularity, we consider the complex $(\Omega_{V,0}^\bullet, \wedge\omega)$:

$$0 \to \mathcal{O}_{V,0} \to \Omega_{V,0}^1 \to ... \to \Omega_{V,0}^n \to 0,$$

where $\Omega_{V,0}^i$ are the modules of sheaves of differential forms on $(V, 0)$ and the arrows are given by the exterior product by the form ω.

This complex is the dual of the Koszul complex considered in [10], (1.4). It was used by G.M. Greuel in [12] for complete intersections. The sheaves $\Omega^i_{V,0}$ are coherent sheaves and the homology groups of the complex $(\Omega^\bullet_{V,0}, \wedge\omega)$ are concentrated at the origin and therefore are finite dimensional. The definition below was inspired by that of [10] for vector fields.

Definition 3.1. The *homological index* $\mathrm{Ind}_{\mathrm{hom}}(\omega; V, 0) = \mathrm{Ind}_{\mathrm{hom}}\omega$ of the 1-form ω on $(V, 0)$ is $(-1)^n$ times the Euler characteristic of the above complex:

$$(1) \qquad \mathrm{Ind}_{\mathrm{hom}}(\omega; V, 0) = \mathrm{Ind}_{\mathrm{hom}}\omega = \sum_{i=0}^{n}(-1)^{n-i}h_i(\Omega^\bullet_{V,0}, \wedge\omega),$$

where $h_i(\Omega^\bullet_{V,0}, \wedge\omega)$ is the dimension of the corresponding homology group as a vector space over $\mathbb{C}$.

Theorem 3.2. *Let ω be a holomorphic 1-form on V with an isolated singularity at the origin 0.*

(i) *If V is smooth, then $\mathrm{Ind}_{\mathrm{hom}}\omega$ equals the usual local index of the 1-form ω.*

(ii) *The homological index satisfies the law of conservation of number: if ω' is a holomorphic 1-form on V close to ω, then:*

$$\mathrm{Ind}_{\mathrm{hom}}(\omega; V, 0) = \mathrm{Ind}_{\mathrm{hom}}(\omega'; V, 0) + \sum \mathrm{Ind}_{\mathrm{hom}}(\omega'; V, x),$$

where the sum on the right hand side is over all those points x in a small punctured neighbourhood of the origin 0 in V where the form ω' vanishes.

(iii) *If $(V, 0)$ is an isolated complete intersection singularity, then the homological index $\mathrm{Ind}_{\mathrm{hom}}\omega$ coincides with the index $\mathrm{Ind}\,\omega$.*

Proof. Statement (i) is straightforward and it is a special case of statement (iii). Statement (ii) is a particular case of the main theorem in [9]. For statement (iii), we notice that (on an isolated complete intersection singularity $(V, 0)$) the index $\mathrm{Ind}\,\omega$ also satisfies the law of conservation of number and coincides with the homological index $\mathrm{Ind}_{\mathrm{hom}}\omega$ on smooth varieties. This implies that the difference between these two indices is a locally constant, and therefore constant, function on the space of 1-forms on $(V, 0)$ with an isolated singular point at the origin. Therefore it suffices to prove (iii) for $\omega = df$ where f is a holomorphic function on $(V, 0)$ with an isolated critical point at the origin. The de Rham lemma in [12], Lemma 1.6, implies that the homology groups of the complex $(\Omega^\bullet_{V,0}, \wedge df)$ vanish in dimensions $i = 0, 1, ..., n-1$. The statement then follows from the Remark after Lemma 5.3 of [12] (see also [7]). $\qquad\square$

REMARK 3.3. The minimal value of the homological index $\mathrm{Ind}_{\mathrm{hom}}\omega$ is attained by restrictions to V of generic 1-forms on $\mathbb{C}^N$ which do not vanish at the origin. The subset of forms with this index in $\Omega^1_{V,0}$ is open, dense and connected. Moreover, each 1-form ω can be approximated by a 1-form, the index of which at the origin coincides with the minimal one and all its zeroes on $V - \{0\}$ are non-degenerate. This approximation can be chosen of the form $\omega + \varepsilon d\ell$ for a linear function ℓ.

REMARK 3.4. We notice that one has an invariant for functions on $(V, 0)$ with an isolated singularity at the origin defined by $f \mapsto \mathrm{Ind}_{\mathrm{hom}} df$. By the theorem above, if $(V, 0)$ is an isolated complete intersection singularity, this invariant counts the number of critical points of the function f on a Milnor fibre.

REMARK 3.5. Let $(C, 0)$ be a curve singularity and let $(\bar{C}, \bar{0})$ be its normalization. Let $\tau = \dim \mathrm{Ker}(\Omega^1_{C,0} \to \Omega^1_{\bar{C},\bar{0}})$, $\lambda = \dim \omega_{C,0}/c(\Omega^1_{C,0})$, where $\omega_{C,0}$ is the dualizing module of Grothendieck, $c : \Omega^1_{C,0} \to \omega_{C,0}$ is the class map (see [4]). In [17] there is considered a Milnor number of a function f on a curve singularity introduced by V.Goryunov. One can see that this Milnor number can be defined for a 1-form ω with an isolated singularity on $(C, 0)$ as well (as $\dim \omega_{C,0}/\omega \wedge \mathcal{O}_{C,0}$) and is equal to $\mathrm{Ind}_{\mathrm{hom}} \omega + \lambda - \tau$.

4. A GENERALIZED MILNOR NUMBER

We recall that, by Proposition 2.8, for 1-forms on an isolated complete intersection singularity the difference between its index and its radial index is the Milnor number of the singularity, independently of the choice of the 1-form. On a germ $(V, 0)$ in general, the index $\mathrm{Ind}\,\omega$ of a 1-form is not defined, but the homological index $\mathrm{Ind}_{\mathrm{hom}}\,\omega$ is and it coincides with the index $\mathrm{Ind}\,\omega$ for complete intersection germs. The radial index is always defined.

The laws of conservation of numbers for the homological and the radial indices of 1-forms together with the fact that these two indices coincide on smooth varieties imply that their difference is a locally constant, and therefore constant, function on the space of 1-forms on V with isolated singularities at the origin. Therefore one has the following statement.

Proposition 4.1. *Let $(V, 0)$ be a germ of a complex analytic space of pure dimension n with an isolated singular point at the origin. Then the difference*

$$\mathrm{Ind}_{\mathrm{hom}}\,\omega - \mathrm{Ind}_{\mathrm{rad}}\,\omega$$

between the homological and the radial indices does not depend on the 1-form ω.

This proposition, together with 2.8, permits to consider the difference

$$\nu(V, 0) = \mathrm{Ind}_{\mathrm{hom}}(\omega; V, 0) - \mathrm{Ind}_{\mathrm{rad}}(\omega; V, 0)$$

as a generalized Milnor number of the singularity $(V, 0)$.

REMARK 4.2. One can define the invariant $\nu(V, 0)$ using any 1-form ω on V with an isolated singular point at the origin, say, the differential df of a holomorphic function f on V. In this case one can describe $\nu(V, 0)$ in somewhat different terms. When $(V, 0)$ is an isolated complete intersection singularity, the homological index $\mathrm{Ind}_{\mathrm{hom}} df$ is equal to the sum $\mu(V, 0) + \mu(f)$ where $\mu(f)$ is the Milnor number of the germ f on $(V, 0)$. Thus for an arbitrary isolated singularity $(V, 0)$ and for a function germ f on it, the homological index $\mathrm{Ind}_{\mathrm{hom}} df$ can be considered as a generalization of this sum. The Milnor number $\mu(f)$ of a germ f of a holomorphic function with an isolated critical point can be defined

for any isolated singularity $(V,0)$ of pure dimension n as $(-1)^n(1-\chi(F))$, where F is the Milnor fibre of f, and is equal to the radial index $\mathrm{Ind}_{\mathrm{rad}}\,df$ (see Example 2.6). Therefore the invariant $\nu(V,0)$ is a generalization of the number $\mu(V,0)+\mu(f)$ minus the Milnor number $\mu(f)$.

There are other invariants of isolated singularities of complex analytic varieties which coincide with the Milnor number for isolated complete intersection singularities. One of them is $(-1)^n$ times the reduced Euler characteristic (i.e., the Euler characteristic minus 1) of the absolute de Rham complex of $(V,0)$. It is natural to try to compare $\nu(V,0)$ with them.

Theorem 4.3. *For a curve singularity $(C,0)$,*

$$\nu(C,0) = \dim_{\mathbb{C}} \Omega^1_{C,0}/d\mathcal{O}_{C,0}\,.$$

Proof. Let $\pi : (\bar{C},\bar{0}) \to (C,0)$ be the normalization of the curve $(C,0)$, $\bar{0} = \pi^{-1}(0)$. Let $C = \cup_{i=1}^{r}C_i$ be the decomposition of the curve $(C,0)$ into the union of irreducible components, let t_i be the uniformization parameter on C_i, and let $\omega_{|C_i} = (a_i t_i^{m_i} +$ *terms of higher degree*$)dt_i$, $a_i \neq 0$. Consider two commutative diagrams with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{C,0} & \xrightarrow{\wedge\omega} & \Omega^1_{C,0} & \longrightarrow & \Omega^1_{C,0}/\omega\wedge\mathcal{O}_{C,0} & \longrightarrow & 0 \\
& & \pi_0^*\downarrow & & \pi_1^*\downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\bar{C},\bar{0}} & \xrightarrow{\wedge\bar{\omega}} & \Omega^1_{\bar{C},\bar{0}} & \longrightarrow & \Omega^1_{\bar{C},\bar{0}}/\bar{\omega}\wedge\mathcal{O}_{\bar{C},\bar{0}} & \longrightarrow & 0
\end{array}
$$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{m}_{C,0} & \xrightarrow{d} & \Omega^1_{C,0} & \longrightarrow & \Omega^1_{C,0}/d\mathcal{O}_{C,0} & \longrightarrow & 0 \\
& & \widetilde{\pi}_0^*\downarrow & & \pi_1^*\downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{m}_{\bar{C},\bar{0}} & \xrightarrow{d} & \Omega^1_{\bar{C},\bar{0}} & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
$$

where $\mathfrak{m}_{C,0}$ is the maximal ideal in the ring $\mathcal{O}_{C,0}$, $\mathfrak{m}_{\bar{C},\bar{0}}$ is the ideal of germs of functions on the normalization $(\bar{C},\bar{0})$ equal to zero at all the point from $\bar{0}$. Note that the homomorphisms π_0^* and $\widetilde{\pi}_0^*$ are injective. Here $\dim \Omega^1_{C,0}/\omega\wedge\mathcal{O}_{C,0}$ is equal to the homological index $\mathrm{Ind}_{\mathrm{hom}}\,\omega$, $\dim \Omega^1_{\bar{C},\bar{0}}/\bar{\omega}\wedge\mathcal{O}_{\bar{C},\bar{0}} = \sum m_i = \mathrm{Ind}_{\mathrm{rad}}\,\omega - (r-1)$ (see Example 2.5), $\dim\mathrm{coker}\,\pi_0^* = \dim\mathrm{coker}\,\widetilde{\pi}_0^* + (r-1)$. Applying the Snake Lemma (see, e.g., [8]) to these diagrams (all kernels and cokernels of the vertical homomorphisms are finite dimensional) we get the statement. $\square$

REMARK 4.4. A notion of a generalized Milnor number of a curve singularity $(C,0)$ was introduced in [4] as $\dim_{\mathbb{C}}\omega_{C,0}/d\mathcal{O}_{C,0}$, where $\omega_{C,0}$ is the dualizing module of Grothendieck. For smoothable curve singularities, it is equal to $1-\chi(\widetilde{C})$, where $\widetilde{C}$ is a smoothing of $(C,0)$. (All smoothings of a curve singularity have the same Euler characteristic.) From the proof of [4], Theorem 6.1.3, it follows that the Milnor number defined by R.-O. Buchweitz and G.-M. Greuel is equal to $\nu(C,0)+\lambda-\tau$, where τ and λ are defined in Remark 3.5. For complete intersection curve singularities $\lambda=\tau$.

EXAMPLE 4.5. Let $(S, 0) \subset (\mathbb{C}^5, 0)$ be the cone over the rational normal curve in $\mathbb{CP}^4$ (this is Pinkham's example [18] mentioned in the Introduction). According to Example 2.6, for a generic linear function ℓ on $\mathbb{C}^5$, the radial index $\mathrm{Ind}_{\mathrm{rad}}\, d\ell$ of its differential on the surface S is equal to the Milnor number $\mu(\ell_{|S})$ (see Remark 4.2 for the definition of the latter one). This Milnor number can be easily computed (say, using the formula of N. A'Campo [1]) and is equal to 3. All the equations of the surface S are homogeneous of degree 2 and therefore the modules $\Omega^i_{S,0}$, $i = 0, 1, 2$, have natural gradings (where we consider the differentials dx_j of the variables having degree 0, $j = 1, \dots, 5$). The differentials in the complex $(\Omega^\bullet_{S,0}, \wedge d\ell)$ respect the grading. The differentials in the absolute de Rham complex $(\Omega^\bullet_{S,0}, d)$ have degree (-1). Therefore the Euler characteristics of these complexes can be computed from the Poincaré series $P_i(t)$ of the gradings on the modules $\Omega^i_{S,0}$. These Poincaré series (computed using MACAULAY) are equal to:

$$
\begin{aligned}
P_0(t) &= 1 + 5t + 9t^2 + 13t^3 + 17t^4 + 21t^5 + 25t^6 + 29t^7 + 33t^8 + 37t^9 + \dots, \\
P_1(t) &= 5 + 19t + 24t^2 + 32t^3 + 40t^4 + 48t^5 + 56t^6 + 64t^7 + 72t^8 + 80t^9 + \dots, \\
P_2(t) &= 10 + 21t + 15t^2 + 19t^3 + 23t^4 + 27t^5 + 31t^6 + 35t^7 + 39t^8 + 43t^9 + \dots.
\end{aligned}
$$

Therefore the Euler characteristic of the complex $(\Omega^\bullet_{S,0}, \wedge d\ell)$ (equal to the homological index $\mathrm{Ind}_{\mathrm{hom}}\, d\ell$) is equal to $(P_0(t) - P_1(t) + P_2(t))_{|t \mapsto 1} = 13$ and the invariant $\nu(S, 0)$ introduced above is equal to 10. One can see that the reduced Euler characteristic of the absolute de Rham complex of $(S, 0)$ is equal to $(P_0(t) - tP_1(t) + t^2 P_2(t))_{|t \mapsto 1}$ and hence also equal to 10.

REMARK 4.6. Note that the differentials of the complex $(\Omega^\bullet_{V,0}, \wedge \omega)$ are homomorphisms of $\mathcal{O}_{V,0}$–modules while those of the de Rham complex do not have this property.

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WOLFGANG EBELING: UNIVERSITÄT HANNOVER, INSTITUT FÜR MATHEMATIK, POSTFACH 6009, D-30060 HANNOVER, GERMANY.
  E-mail address: ebeling@math.uni-hannover.de

SABIR M. GUSEIN-ZADE: MOSCOW STATE UNIVERSITY, FACULTY OF MECHANICS AND MATHEMATICS, MOSCOW, GSP-2, 119992, RUSSIA.
  E-mail address: sabir@mccme.ru

JOSÉ SEADE: INSTITUTO DE MATEMÁTICAS, UNIDAD CUERNAVACA, UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO, APARTADO POSTAL 273-3, C.P. 62210, CUERNAVACA, MORELOS, MÉXICO.
  E-mail address: jseade@matem.unam.mx